Efficient Calculation of Optimum Design Sensitivity

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Two methods for obtaining the sensitivity of an optimized design to changes in problem parameters are presented. The first-order method uses gradient information to estimate the required sensitivity. The second-order method uses second derivatives of the design objective and constraint functions to provide a quadratic approximation to the new design that results from changing the specified parameter. It is shown that the sensitivity of the optimized design may be discontinuous with respect to the new parameter. This discontinuity is accounted for by the present methods. These methods are compared to a previous approach based on the Kuhn-Tucker necessary conditions for optimality. It is shown that the present methods provide the appropriate sensitivity information in an efficient manner. The methods are demonstrated by a structural synthesis example.

Nomenclature

A_i	= cross-sectional area of member i
c	= constant
$\mathrm{d}X$	= perturbation vector
$\mathrm{d}X_i$	= ith component of $\mathrm{d}X$
$F(X)$	= objective function
$g_j(X)$	= jth inequality constraint
H_F	= Hessian matrix of the objective function
H_j	= Hessian matrix of the jth constraint
J	= set of critical and near-critical constraints
n	= number of design variables
m	= number of inequality constraints
P	= independent parameter to be changed
X	= vector of design variables
X_i	= ith component of X
X_i^l, X_i^u	= lower and upper bounds on X_i
S	= search direction
S_i	= ith component of S
α	= move parameter in the one-dimensional search
β	= intermediate variable
δP	= change in the independent parameter P
δX	= change in X
∇	= gradient operator
λ	= Lagrange multiplier
$\bar{\sigma}$	= allowable stress

Introduction

ESTIMATION of the sensitivity of an optimum design to some new problem parameter P (or perhaps multiple parameters) has been the subject of considerable recent research. There are two principal reasons for needing this information. First, if the problem requirements are modified after the optimization is complete, this will provide the engineer with a measure of what effect such changes will have on the design; and second, this information is directly useful in formal multilevel and multidiscipline optimization.

The general optimization problem considered here is of the following form.

Find the set of design variables X^* that will

$$\text{Minimize } F(X) \tag{1}$$

Subject to:

$$g_j(X) \le 0 \qquad j=1,m \tag{2}$$

$$X_i^l \le X_i \le X_i^u \qquad i=1,n \tag{3}$$

In addition to the inequality constraints of Eq. (2), equality constraints can be included also. Equality constraints are omitted here only for brevity, and their addition is a direct extension of the methods presented here.

Assume the optimization problem has been solved, so the vector of optimum design variables X^* is known. Now assume some new parameter P is to be changed, where P may be a load, allowable stress, material property, or any other problem parameter. It is desired to determine how the optimum design will change as a result of changing P. That is, we wish to find the total derivative $\mathrm{d}F(X^*)/\mathrm{d}P$ as well as the rates of change of the optimum values of the design variables themselves, $\partial X_i/\partial P$.

In Refs. 1-4, the Kuhn-Tucker conditions at the optimum are used to predict the required derivatives, based on the assumption that the Kuhn-Tucker conditions at X^* remain in force as P is changed. Normally, second derivatives of the objective and binding constraints are required, as well as the Lagrange multipliers associated with the optimum design.

In Ref. 5, a method was introduced based on the concept of a feasible direction, for providing the optimum sensitivity information. This method requires only first derivatives of the objective function and binding constraints.

Here, the method of Ref. 5 is expanded to deal effectively with the possibility of a discontinuous derivative at X^*. This first-order method provides the sensitivity information in the classical sense of a derivative. A second-order method is presented here for use in those design situations where second derivatives are economically available. In this case an approximate optimization problem is actually performed to estimate the design improvement that can be expected from a change in the new parameter.

The three available approaches to optimum design sensitivity are compared using simple examples to gain a conceptual understanding of the methods. The present method is then demonstrated by physical example.

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Sensitivity Using the Kuhn-Tucker Conditions

At the optimum X^*, some subset J of constraints will be critical. The Kuhn-Tucker conditions[6] that are satisfied at this point are

$$\nabla F(X^*) + \sum_{j=1}^{m} \lambda_j \nabla g_j(X^*) = 0 \tag{4}$$

$$g_j(X^*) = 0 \qquad j \in J \tag{5}$$

$$\lambda_j \geq 0 \tag{6}$$

The needed sensitivity information comes from calculating the total derivative of $F(X^*)$ with respect to the new parameter P,

$$\frac{\mathrm{d}F(X^*)}{\mathrm{d}P} = \frac{\partial F(X^*)}{\partial P} + \nabla F(X^*) \cdot \mathrm{d}X \tag{7}$$

where

$$\mathrm{d}X_i = \frac{\partial X_i}{\partial P} \tag{8}$$

If the Kuhn-Tucker conditions of Eqs. (4-6) are to remain satisfied for some change δP in the independent parameter P, then the rate of change of Eq. (4) with respect to P must vanish, where it is noted that Eq. (4) is actually n independent equations. This leads to a set of simultaneous equations that are solved for dX as well as dλ (since the Lagrange multipliers are functions of P as well). The coefficient matrix of these equations contains the second derivatives of the objective and binding constraints with respect to X_i, $i = 1, n$, and P. The two significant features of this approach are that second-order information is required and that constraints that are critical at X^* remain critical when the independent parameter P is changed.

Sensitivity Using the Feasible Directions Concept

Conceptually, calculation of the sensitivity to parameter P can be viewed as seeking the greatest improvement in the expanded design space that includes the new "design variable" P. Thus, we mathematically seek the "constrained steepest-descent direction." Assuming we are free to either increase or decrease P as necessary to improve the design, the information is obtained from the following subproblem.

Treat P as an independent design variable and add it to the set of variables X, so that

$$X_{n+1} = P \tag{9}$$

Now solve the following direction-finding problem in the expanded space; find the components of S to

$$\text{Minimize } \nabla F(X^*) \cdot S \tag{10}$$

Subject to:

$$\nabla g_j(X^*) \cdot S \leq 0 \qquad j \in J \tag{11}$$

$$S \cdot S \leq 1 \tag{12}$$

Equation (10) represents the objective, which is to search in a direction as nearly possible to the steepest-descent direction. This is constrained, however, by Eq. (11), which dictates that the search direction be tangent to, or away from, the boundaries of the critical constraints. Note that by virtue of the inequality condition of Eq. (11), the design may actually leave a constraint boundary if this will give maximum improvement. This is equivalent to saying that the Kuhn-Tucker conditions in force at X^* need not remain in force in the expanded design space. Equation (12) is required to ensure a unique solution to the direction-finding process.

This geometric interpretation of a constrained steepest-descent direction is also reached by considering a linear approximation to the problem in expanded space as: Find the perturbation of the design variables contained in the vector S that will

$$\text{Minimize } \tilde{F}(X,P) = F(X^*,P) + \nabla F(X^*,P) \cdot S \tag{13}$$

Subject to:

$$\tilde{g}_j(X,P) = g_j(X^*,P) + \nabla g_j(X^*,P) \cdot S \leq 0 \qquad j \in J \tag{14}$$

$$S \cdot S \leq 1 \tag{15}$$

where Eq. (15) arises from the need to bound the solution to the linearized problem. Noting that $F(X^*,P)$ is constant and $g_j(X^*,P) = 0$ for $j \in J$, it is clear that Eqs. (13-15) are the same as Eqs. (10-12).

Equations (10-12) represent a linear problem in S_i, $i = 1$, $n+1$ subject to a single quadratic constraint. This is solved

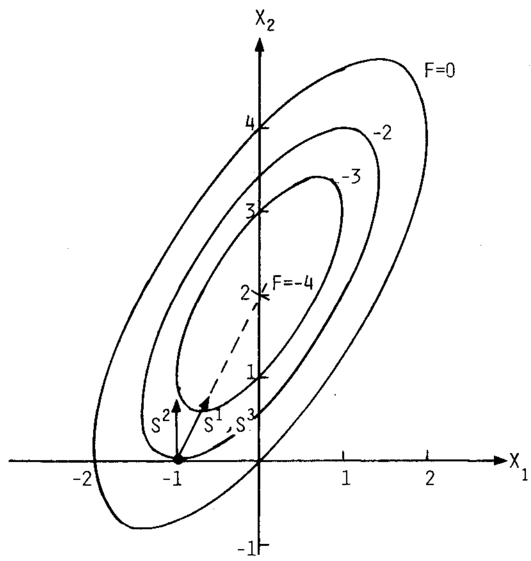

Fig. 1 Unconstrained problem.

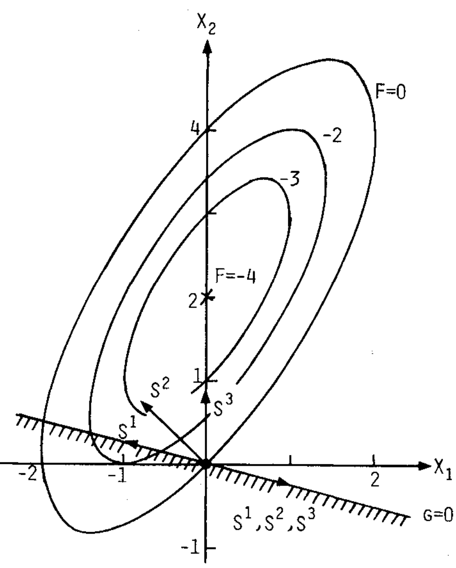

Fig. 2 Discontinuous sensitivities.

by conversion to a linear programming-type problem of dimension $J+1$ and is easily solved numerically.[7]

Having solved the direction-finding problem of Eqs. (10-12), the design can be updated by the common relationship

$$X = X^* + \alpha S \tag{16}$$

so

$$X_i = X_i + \alpha \frac{\partial X_i}{\partial \alpha} \tag{17}$$

and

$$\frac{\partial X_i}{\partial \alpha} = S_i \tag{18}$$

Considering the $n+1$ component P,

$$P = P^* + \alpha \frac{\partial P}{\partial \alpha}$$

$$= P^* + \alpha S_{n+1} \tag{19}$$

For a specified change, $\delta P = P - P^*$, the move parameter α is

$$\alpha^* = \delta P / S_{n+1} \tag{20}$$

In the event that $S_{n+1} = 0$, this indicates that the optimum design is not dependent on P and so P can be changed arbitrarily. In this case, $\mathrm{d}F(X^*) = 0$ and $\mathrm{d}X = 0$.

Here it is important that the sign on δP be the same as the sign on S_{n+1} since it was assumed that we will change P in the direction of maximum improvement.

The rate of change of the optimum objective is now found for a unit change in P as

$$\frac{\mathrm{d}F(X^*)}{\mathrm{d}P} = \frac{\nabla F(X^*) \cdot S}{S_{n+1}} \tag{21}$$

and the corresponding rate of change of the optimum design variables is

$$\mathrm{d}X = (1/S_{n+1}) S \tag{22}$$

Multiplying Eqs. (21) and (22) by δP gives the estimated change in $F(X^*)$ and X^*, respectively. If δP is of the opposite sign from S_{n+1}, Eqs. (21) and (22) still apply if Eq. (11) is satisfied with strict equality. If this is not the case, the design dictated here will leave a constraint for maximum improvement. Thus, a search in the opposite direction will violate one or more constraints.

Consider now the case where the sign of the change in parameter P is specified. Now it must be recognized that if P is decreased, the optimum design may follow one constraint surface, but if P is increased, the design may follow a different constraint surface. In other words, the rate of change of the optimum design with respect to P may not be constant at X^*. In this case, an additional constraint must be imposed on the sign of S_{n+1}. For example, if P is required to be positive, the direction-finding problem of Eqs. (10-12) becomes

$$\text{Minimize } \nabla F(X^*) \cdot S - c\beta \tag{23}$$

Subject to:

$$\nabla g_j(X^*) \cdot S \leq 0 \tag{24}$$

$$-S_{n+1} + \beta \leq 0 \tag{25}$$

$$S \cdot S \leq 1 \tag{26}$$

The independent variables in this direction-finding problem are the components of S as well as the extra parameter β. Here the constant c is a somewhat arbitrary positive number, say, 1000. The magnitude of c is not critical since it is used only to ensure that the resulting value of β will be positive and so S_{n+1} will in turn be positive. If δP is specified to be negative, Eq. (25) is replaced by

$$S_{n+1} + \beta \leq 0 \tag{27}$$

The result of the direction-finding problem is to find a search direction that will increase $F(X^*)$ as little as necessary, decreasing $F(X^*)$ if possible. Thus, if there is a specific reason to change P in a given direction, the optimum sensitivity is still found.

The method of Eqs. (23-26) is actually the same as that used in the method of feasible directions for overcoming constraint violations.[7]

The method of Eqs. (10-12) and (23-26) is based on the assumption that P will be changed to gain maximum improvement (or minimum degradation) in the optimum objective. The only additional information required beyond that normally available is the gradient of the objective and critical constraints with respect to P. Also, it is important to note that the set J can include any near-critical constraints that we do not wish to become violated for a small change in P. Finally, it should be noted that a search direction may be desired that moves away from the constraint boundaries to give a more conservative estimate of $\mathrm{d}F(X^*)/\mathrm{d}P$. This can be accomplished by adding a push-off factor to Eq. (11) or (24), as is done in the conventional feasible directions algorithm.[7]

Sensitivity Using Second-Order Information

In cases where second derivatives are available, this information should be used to provide maximum guidance to the design process. Here, the information sought is not sensitivity in the mathematical sense, but rather the best estimate of the new optimum based on a quadratic approximation to the objective and constraint functions about X^*.

Consider a second-order Taylor series expansion about X^* of $F(X^*)$ and $g_j(X^*)$ $j \epsilon K$, where K includes the set of critical and near-critical constraints (and could include the entire set m). Here also, we expand the set of design variables so $X_{n+1} = P$ as before.

The approximate optimization task now becomes, find the change in design variables δX to

$$\text{Minimize } \tilde{F}(X) = F(X^*) + \nabla F(X^*) \cdot \delta X + \tfrac{1}{2} \delta X^T H_F \delta X \tag{28}$$

Subject to:

$$\tilde{g}_j(X) = g_j(X^*) + \nabla g_j(X^*) \cdot \delta X + \tfrac{1}{2} \delta X^T H_j \delta \leq 0 \tag{29}$$

$$\delta X_i^l \leq \delta X_i \leq \delta X_i^u \qquad i = 1, n+1 \tag{30}$$

where

$$\delta X = X - X^* \tag{31}$$

Here, P is treated as an independent variable, and the move limits of Eq. (30) are imposed to limit the search to the region of validity of the quadratic approximations. In the event that δP (and, therefore, the new value of P) is specified, then the optimization is carried out with respect to the original n variables, with $\delta X_{n+1} = \delta P$.

The problem of Eqs. (28-30) requires the same information as sensitivity based on the Kuhn-Tucker conditions (except the Lagrange multipliers are not needed here) but is complicated by the fact that a constrained optimization task is now required. However, the functions are now explicit and easily evaluated, along with their derivatives. The advantage here is that the approximate optimization task accounts for the complete set of constraints K and that no limiting assumptions are needed regarding the nature of the new optimum. Also, in the common situation in structural optimization using reciprocal variables, where the constraints are approximately linear, only the linear portion of Eq. (29) is needed, greatly simplifying the optimization task.

Comparison of Methods

In the previous sections, three distinct approaches were given for determining the sensitivity of an optimized design to some new parameter P based on 1) the Kuhn-Tucker conditions at X^*; 2) the feasible direction concept; and 3) second-order expansion of the problem.

These three approaches are compared here using simple examples to demonstrate geometrically the similarities and differences in the methods. Each method can be considered to produce a search direction S^q, where $q=1,2,3$ corresponds to the particular method. The total derivative is proportional to S^q, so any differences in the methods are reflected in differing search directions.

Case 1—Unconstrained Function

Consider the simple two-variable unconstrained problem,

$$\text{Minimize } F=2X_1^2-2X_1P+P^2+4X_1-4P \tag{32}$$

While this is an explicit example function, it could as well represent the second-order approximation to a far more complicated problem.

The problem is first solved with respect to the single variable X_1, and then sensitivity is calculated with respect to the new parameter P.

Figure 1 is the two-variable function space for this problem, where initially $P=0$, so the optimum with respect to X_1 is $F^*=-2$ at $X_1=-1$. The search directions for the three methods are shown on the figure, and it is seen that methods 1 and 3, using second derivatives, point to the solution of the quadratic problem. Method 2, being a first-order method, gives a steepest-descent search direction.

Case 2—Constrained Problem with Discontinuous Derivatives

Figure 2 shows the results for the same problem with the addition of the linear inequality constraint,

$$g=-X_1-4P\le 0 \tag{33}$$

Now minimization with respect to X_1 gives the constrained optimum, $F=0$, $g=0$ at $X_1=0$. Here the three approaches provide markedly different information. Method 1 produces a search direction that follows the constraint for either an increase or a decrease in P. If P is increased, method 2 gives a direction of steepest descent, while method 3 provides a direction toward the quadratic approximation to the minimum. If P is changed in the negative direction, each method results in a move along the constraint boundary. Clearly, because the optimum change direction (whether first or second order) is dependent on the sign of δP, the total derivative of the optimum objective is discontinuous at X^*.

Case 3—Multiple Constraints

Here the constrained optimization problem to be solved is

$$\text{Minimize } F=X_1^2+(P-1)^2 \tag{34}$$

Subject to:

$$g_1=-3X_1-2P+10\le 0 \tag{35}$$

$$g_2=-2X_1-3P+10\le 0 \tag{36}$$

The problem is first solved with P held fixed at a value of 2. Then the sensitivity of the optimum to P is calculated. The two-variable function space and the calculated search directions are shown in Fig. 3. This is a degenerate case, which cannot be solved by method 1 due to matrix singularity. Here at X^*, two constraints are critical. However, because the problem is first a function of X_1 alone, one of these constraints is redundant. Methods 2 and 3 provide change vectors that follow the proper constraint boundary, depending on the sign of δP.

Case 4—Dependence on the Magnitude of P

Here the problem to be solved is

$$\text{Minimize } F=X_1^2+P^2 \tag{37}$$

Subject to:

$$g=(X_1-5)^2+(P-5)^2-16\le 0 \tag{38}$$

The two-variable function space and the calculated search directions are shown in Fig. 4.

Here, methods 1 and 2 provide the same search direction toward point D. Method 3, however, takes full advantage of the available second-order information to identify the new optimum based on the quadratic approximation. If P is allowed to change freely, the new optimum is at point B. This is the usual result for method 3 if no move limits are imposed on the design changes and the actual amount by which P is changed is not specified. However, if the change δP is specified, method 3 provides the best approximation to the new optimum with respect to the original design variable X_1 at point C. Here, it is important to remember that this result is based on a quadratic approximation to the original problem so, in practice, move limits will be required to ensure reasonable bounds on the solution. It is noteworthy that methods 1 and 2 provide the same search direction. This is because method 1, while requiring second derivatives of the constraints, does not actually use this information to update the approximation to the optimum design.

Consistency of the Sensitivity Calculations

Because the sensitivity is calculated about a numerically determined optimum, it is expected that the results will be dependent on the accuracy of X^*. It is common that, while the objective function may be very near the theoretical optimum, the design vector X^* is not this precise. Also, the

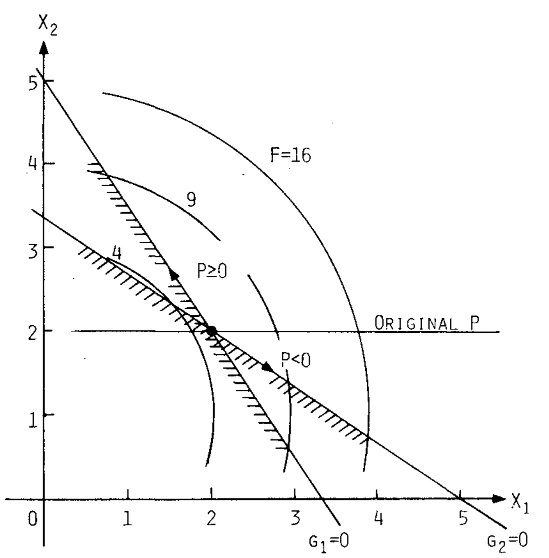

Fig. 3 Multiple constraints.

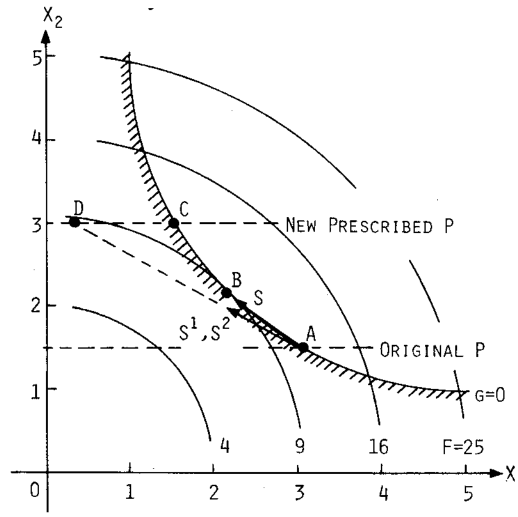

Fig. 4 Specified change in *P*.

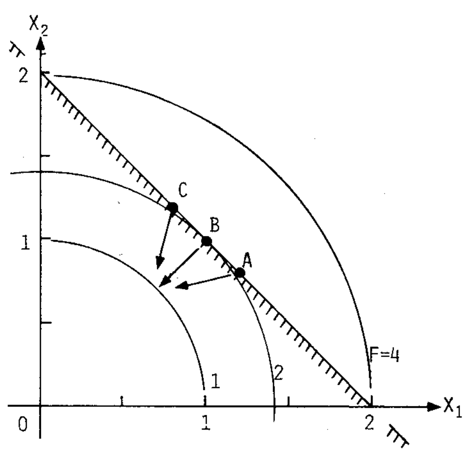

Fig. 5 Sensitivity for different optima.

Table 1 Sensitivities for various optima

Variable	X^A	S^A	X^B	S^B	X^C	S^C
X_1	1.2	−0.8	1.0	−0.5	0.8	−0.2
X_2	0.8	−0.2	1.0	−0.5	1.2	−0.8
F^*	2.08	—	2.00	—	2.08	—
dF^*/dP	—	−2.24	—	−2.00	—	−2.24

gradients of the objective and constraint functions at the optimum are more variable in direction than in magnitude. These observations lead to the conclusion that the total derivative of $F(X^*)$ with respect to P is reasonably stable but that the rates of change of the independent design variables are more strongly dependent on the accuracy of the optimum.

This variability of the design vector sensitivities is seen from a simple example using linear sensitivity information.

Figure 5 shows the two-variable function space for the following problem:

$$\text{Minimize } F = X_1^2 + X_2^2 \tag{39}$$

Subject to:

$$g = -X_1 - X_2 + P \leq 0 \tag{40}$$

where initially $P=2$. Points A, B, and C are each near the optimum, with B being the precise optimum. The sensitivity of each "optimum" is shown by the search vectors and given numerically in Table 1.

Thus, it is clear that the design sensitivity is dependent on the accuracy of X^*. However, the information provided is still as useful, being the constrained steepest-descent direction. This simply serves as a reminder that, if the design space is reasonably flat, within numerical accuracy, the optimum design and its sensitivity are not unique.

Design Examples

Figure 6 is the 10-bar truss that is commonly used to demonstrate optimization procedures. The structure is loaded as shown by a single loading condition and is stress-constrained. The allowable stress in each member is 172.4 MPa, with the exception of member 9, which has a higher allowable stress. The cross-sectional areas of the members are the design variables and the total weight of the structure is to be minimized.

Case 1—Comparison of Methods

Table 2 gives the optimum design and sensitivity results for a nominal design in which the allowable stress in member 9 is 206.9 MPa and will be changed in order to improve the optimum. The parameters of interest are listed in the first column, and column 2 gives the initial optimum design. Columns 3 and 4 give the sensitivity based on the Kuhn-Tucker conditions and the present linear method, respectively. For this design, member 10 is very near its stress limit, but the actual binding constraint is the minimum size of the member. Thus, based on the Kuhn-Tucker conditions, the sensitivity of member 10 (as well as members 2, 5, and 6) is zero. In the linear method, this stress constraint is included

Table 2 Sensitivity of the optimum to the allowable stress in member 9

(1) Variables	(2) X^*	(3) $S^{\text{K-T}}$	(4) S^L	(5) X^Q	(6) S^Q	(7) $X^{\text{K-T}}$	(8) X^L	(9) X^Q	(10) X^*
A_1	51.156	−0.3174	−0.7710	50.98	−0.6394	51.05	50.90	50.94	50.97
A_2	0.6452	0.000	0.0000	0.645	0.0000	0.645	0.645	0.645	0.645
A_3	52.067	0.3219	0.7742	52.23	0.5697	52.17	52.32	52.26	52.26
A_4	25.353	−0.3200	−0.7703	25:20	−0.6439	25.25	25.10	25.14	25.16
A_5	0.6452	0.0000	0.0000	0.645	0.0000	0.645	0.645	0.645	0.645
A_6	0.6452	0.0000	0.0000	0.645	0.0000	0.645	0.645	0.645	0.645
A_7	37.139	0.4555	1.0961	37.38	0.8877	37.29	37.50	37.44	37.41
A_8	35.849	−0.4503	−1.0910	35.77	−0.2697	35.70	35.48	35.76	35.58
A_9	29.878	−30.250	−30.786	23.41	23.774	19.79	19.61	21.95	23.72
A_{10}	0.6452	0.0000	1.1019	0.929	1.0329	0.645	1.013	0.987	0.910
$\bar{\sigma}_9$	206.9	1.0000	1.0000	263.2	1.0000	275.9	275.9	275.9	275.9
F^*	700.8	—	—	678.8	—	664.5	665.0	673.6	679.5
$dF/d\bar{\sigma}$	—	−109.1	−108.1	—	−81.0	—	—	—	—

N.B.: A_i is measured in cm^2; F^* in kg; $\bar{\sigma}_9$ in MPa; and $i = 1{,}10$.

Table 3 Discontinuity in sensitivity

Variable	X^*	S^L	X^L	X^*
A_1	50.973	−0.6329	50.931	51.067
A_2	0.6452	1.3181	0.7329	0.6452
A_3	52.257	1.0026	52.324	52.193
A_4	25.161	−0.6213	25.157	25.225
A_5	0.6452	0.0000	0.6452	0.6452
A_6	0.6452	1.4832	0.7439	0.6452
A_7	37.407	1.3394	37.496	37.321
A_8	35.584	−0.9690	35.519	35.674
A_9	23.736	23.455	25.300	25.481
A_{10}	0.9116	1.7090	1.0258	0.8219
$\bar{\sigma}_9$	258.6	−1.0000	241.4	241.4
F^*	679.5	—	658.8	685.4
$dF/d\bar{\sigma}_9$	—	97.84	—	—

N.B.: A_i is measured in cm^2; F^* in kg; $\bar{\sigma}$ in MPa; and $i=1,10$.

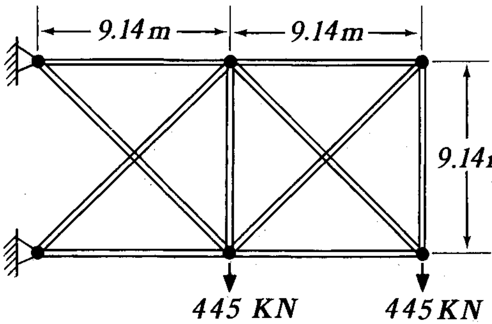

Fig. 6 Ten-bar truss.

in the active set, and the method recognizes that the size of member 10 must be increased to maintain feasibility.

The total derivative $dF/d\bar{\sigma}_9$ is given for a 100% change in $\bar{\sigma}_9$, so the predicted change in F^* is the total derivative times the fractional change to be made in $\bar{\sigma}_9$.

Column 5 gives the proposed optimum X vector if the allowable stress in member 9 is allowed to increase arbitrarily and column 6 gives the resulting sensitivity vector. It is noteworthy that this method did not increase the allowable stress without bound, but in fact recognized the diminishing benefit of increasing this parameter. Indeed, an increase beyond 258.6 MPa provides no additional improvement and the prediction of an "optimum" allowable stress of 263.2 is quite close to this, considering that the sensitivity information is calculated at 206.9 MPa.

Columns 7-9 of Table 2 give the predicted design for each method for an allowable stress in member 9 of 275.9 MPa (a 33% change from its nominal value), and column 10 gives the calculated optimum. The first two methods predicted essentially the same optimum, except that the present linear method accounted for the need to increase the size of member 10. The quadratic method provided the result nearest the calculated optimum, as would be expected.

Case 2—Discontinuity of the Sensitivity

Linear sensitivity of the 10-bar truss was calculated here for a nominal value of 258.6 MPa for the allowable stress in member 9. Now the direction of change in the allowable stress was specified and the resulting problem was solved by the method of Eqs. (23-27). If the direction of change in the allowable stress was not specified or was specified to be positive, the resulting sensitivities were zero, correctly indicating that the optimum weight cannot be reduced by changing $\bar{\sigma}_9$.

Table 3 gives the resulting sensitivities if the allowable stress is required to be reduced. It is noteworthy that the projected optimum for an allowable stress of 241.4 MPa is very near the calculated optimum, even though the projected values of the design variables are not this precise. This again underscores the somewhat nonunique nature of the optimum sensitivity calculations.

This example clearly demonstrates the need to account for the possibility that the design sensitivity may be discontinuous at X^*. In practice, this discontinuity cannot be identified in any a priori fashion, and so the mathematics of the procedure must be relied on to deal with that situation.

Summary

Two general procedures for calculating the sensitivity of an optimized design to some problem parameter have been presented and demonstrated. The linear method provides the sensitivity in the classical sense of a derivative, while accounting for the inequality constrained nature of design. The quadratic method is more appropriately considered an improved estimate of the optimum using available second-order information. These methods have been compared to a previous method based on the Kuhn-Tucker conditions and have been shown to be competitive as well as properly accounting for the possible discontinuity of the sensitivities.

In practice, careful problem formulation can be expected to improve the quality of the projected optimum. For example, for many structural optimization problems, the use of reciprocal variables will allow much larger perturbations in the design without significant constraint violations.

As noted in the Introduction, sensitivity information is valuable in its own right for estimating the effect that design changes will have on the optimum, without reoptimization. Additionally, this capability provides a convenient tool for use in multilevel and multidiscipline design, particularly where distributed computing is desirable. This general area of study is expected to be the direction of future research using these techniques.

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